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Nonlocal symmetry generators and explicit solutions of some partial differential equations

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Abstract

The nonlocal symmetry of a partial differential equation is studied in this paper. The partial differential equation written as a conservation law can be transformed into an equivalent system by introducing a suitable potential. The nonlocal symmetry group generators of original partial differential equations can be obtained through their equivalent system. Further, new explicit solutions can be constructed from the newly obtained symmetry generators. The Burgers equation is chosen as an example; many new valuable explicit solutions and nonlocal symmetry generators are presented.

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1. Introduction

The symmetry group, introduced by S Lie into the study of differential equations, is an effective and systematic technique in handling partial differential equations (PDEs). The classical symmetry group of partial differential equations is the largest local group of transformation acts on the space of independent and dependent variables with the property that it maps solutions of partial differential equation into other solutions. The symmetry group is frequently used in reducing complex PDEs, seeking conservation laws and constructing explicit solutions of PDEs. The local symmetry group of PDEs can be obtained by requiring that the original PDEs keep invariant under group transformation. The general method used to calculate the symmetry group and construct explicit solutions from the symmetry group generators to PDEs is demonstrated systematically in [1–4]. It should be pointed out that one can use symmetry groups to find invariant solutions, but the use of symmetries yields more than just invariant solutions. For example P Olver discusses how to use symmetry groups to obtain solutions in chapter 2, and invariant solutions in chapter 3, of [1].

The potential symmetry of partial differential equations was considered by Bluman *et al* [5, 6]. The potential symmetry generators to PDEs can be obtained by the following step-by-step method: (1) Attach the invariant surface conditions [7, 8] to undetermined symmetry group generators, (2) derive potential symmetry generators determining equations by requiring that the original PDEs remain invariant under the symmetry transformation, (3) solve potential symmetry generators from their determining equations. By solving these potential symmetry generator determining equations, there exists a possibility for one to find new classes of symmetry generators for the given PDEs. These new symmetry generators are neither classical Lie symmetry group generators nor Lie–Bäcklund transformation. They are no longer local symmetry generators, but nonlocal symmetry generators. All local symmetry group of PDEs can be determined by the Lie algorithm, but no corresponding algorithm exists which can be used to find all nonlocal symmetry of a given PDE.

For the sake of finding the nonlocal symmetry generators of partial differential equations as a conservation law, some new undetermined auxiliary functions are introduced unavoidably. This results in two disadvantages. One is the tedious computation in solving symmetry group generators from their determining equations. The other is the difficulty of seeking explicit solutions from these newly obtained symmetry group generators. In this paper, some appropriate amelioration has been made. In order to lessen the calculations in the process of seeking nonlocal symmetry generators, we require that some symmetry infinitesimal generator coefficient functions do not depend on the added unknown functions explicitly. In the course of constructing explicit solutions from the newly obtained symmetry generators, only part of the characteristic equation is chosen from the symmetry infinitesimal generator. In what follows, the Burgers equation is used as an example to illustrate our work. Vinogradov and Krasil'shchik find a nonlocal symmetry for the Burgers equation, and they deduce the Hopf–Cole transformation by using invariant solutions in [9]. This is the very first paper in which solutions invariant under nonlocal symmetries are considered and shown to be useful. A rigorous geometric theory of nonlocal symmetries has been developed by Krasil'shchik and Vinogradov (see chapter 6 of [10]).

2. Nonlocal symmetry generators and explicit solutions

In this section, we study the nonlocal symmetry of the Burgers equation of the following form:

$$u_t + uu_x - u_{xx} = 0. \tag{1}$$

The associated equivalent system to equation (1) is

$$\begin{cases} v_x = u, \\ v_t = u_x - \dfrac{u^2}{2}. \end{cases} \tag{2}$$

Substituting the first equation of system (2) into the second one, it follows

$$v_t + \frac{v_x^2}{2} - v_{xx} = 0, \tag{3}$$

which is called the adjoint integral equation of equation (1). We use nonlocal symmetry generators of equation (1) to seek new explicit solutions. To this end, let

$$V = \tau(x,t,v)\frac{\partial}{\partial t} + \xi(x,t,v)\frac{\partial}{\partial x} + \phi(x,t,u,v)\frac{\partial}{\partial u} + \eta(x,t,v)\frac{\partial}{\partial v} \tag{4}$$

be a symmetry group generator depending on the independent variables x and t and the dependent variables u and v. We need to determine all possible coefficient functions τ, ξ, ϕ and η. The corresponding one-parameter group $\exp(\varepsilon V)$ is a symmetry group of system (2). It should be pointed out that, in order to find potential or nonclassical potential symmetries of the equation (1), symmetry generator (4) is usually replaced by

$$V^* = \tau(x,t,u,v)\frac{\partial}{\partial t} + \xi(x,t,u,v)\frac{\partial}{\partial x} + \phi(x,t,u,v)\frac{\partial}{\partial u} + \eta(x,t,u,v)\frac{\partial}{\partial v}.$$

This will increase the difficulty of seeking symmetry generators from their determining equations and constructing explicit solutions from the newly obtained symmetry generators. According to the classical symmetry group theory, the symmetry determining equations for system (2) are

$$\begin{cases} \eta_x + v_x\eta_v - v_t(\tau_x + v_x\tau_v) - v_x(\xi_x + v_x\xi_v) = \phi, \\ \eta_t + v_t\eta_v - v_t(\tau_t + v_t\tau_v) - v_x(\xi_t + v_t\xi_v) = \phi_x \\ \quad + u_x\phi_u + v_x\phi_v - u_t(\tau_x + v_x\tau_v) - u_x(\xi_x + v_x\xi_v) - u\phi. \end{cases} \tag{5}$$

Since we are interested in finding explicit solutions which are invariant under the one-parameter group $\exp(\varepsilon V)$, we combine equation (1), the associated, equivalent system (2), the adjoint integral equation (3) and invariant surface conditions $\{\tau u_t + \xi u_x = \phi, \tau v_t + \xi v_x = \eta\}$; it follows

$$\begin{cases} u_x = \dfrac{u^2}{2} + \dfrac{1}{\tau}(\eta - u\xi), \\ u_t = \dfrac{\phi}{\tau} - \dfrac{\xi}{\tau}\left[\dfrac{u^2}{2} + \dfrac{1}{\tau}(\eta - u\xi)\right]. \end{cases} \tag{6}$$

Substituting (6) into the first one of equations (5), we obtain

$$\phi = \eta_x - \frac{\eta}{\tau}\tau_x + u\left(\eta_v - \xi_x + \frac{\xi}{\tau}\tau_x - \frac{\eta}{\tau}\tau_v\right) + u^2\left(\frac{\xi}{\tau}\tau_v - \xi_v\right). \tag{7}$$

Substituting (6) and (7) into the second equation of equations (5), we calculate

$$\begin{aligned} \eta_t + \frac{\eta}{\tau}(\eta_v - \tau_t) - \frac{u\xi}{\tau}(\eta_v - \tau_t) - \frac{1}{\tau^2}(\eta^2 - 2u\xi\eta + u^2\xi^2)\tau_v - u\xi_t = C_x + uB_x + u^2A_x \\ + \left[\frac{u^2}{2} + \frac{1}{\tau}(\eta - u\xi)\right](B + 2uA) + u(C_v + uB_v + u^2A_v) - u(C + uB + u^2A) \\ - \left\{\frac{C + uB + u^2A}{\tau} - \frac{\xi}{\tau}\left[\frac{u^2}{2} + \frac{1}{\tau}(\eta - u\xi)\right]\right\}(\tau_x + u\tau_v) \\ - \left[\frac{u^2}{2} + \frac{1}{\tau}(\eta - u\xi)\right]\xi_x - \frac{u^3}{2}\xi_v \end{aligned} \tag{8}$$

with the functions $A = \frac{\xi}{\tau}\tau_v - \xi_v$, $B = \eta_v - \xi_x + \frac{\xi}{\tau}\tau_x - \frac{\eta}{\tau}\tau_v$ and $C = \eta_x - \frac{\eta}{\tau}\tau_x$. Since the coefficients of the algebraic quadratic equation (8) is independent of u, equating the coefficients of u's powers on both sides of equation (8) and using the arbitrariness of u, we obtain

$$\begin{cases} A_v - \dfrac{\tau_v}{\tau}A + \dfrac{A}{2} = 0, \\ A_x + B_v - \dfrac{2\xi}{\tau}A - \dfrac{A}{\tau}\tau_x + \dfrac{\xi}{2\tau}\tau_x - \dfrac{B}{\tau}\tau_v - \dfrac{\xi_x}{2} - \dfrac{B}{2} = 0, \\ \dfrac{\xi}{\tau}(\eta_v - \tau_t) - \dfrac{\xi\eta}{\tau^2}\tau_v + \xi_t + B_x + C_v + \dfrac{2\eta}{\tau}A \\ \quad - \dfrac{\xi}{\tau}B - \dfrac{\xi^2}{\tau^2}\tau_x - \dfrac{B}{\tau}\tau_x - \dfrac{\tau_v}{\tau}C + \dfrac{\xi}{\tau}\xi_x - C = 0, \\ \eta_t + \dfrac{\eta}{\tau}(\eta_v - \tau_t) - \dfrac{\eta^2}{\tau^2}\tau_v - C_x - \dfrac{\eta}{\tau}B + \dfrac{C}{\tau}\tau_x - \dfrac{\xi\eta}{\tau^2}\tau_x + \dfrac{\eta}{\tau}\xi_x = 0. \end{cases} \tag{9}$$

Solving the symmetry generator coefficient functions τ, ξ and η from equations (9) and substituting them into (7) to calculate the coefficient function ϕ, we can obtain the nonlocal symmetries of equation (1). It seems to be more difficult to solve equations (9) than equation (1) itself, but some special solutions can be derived. We set the symmetry generator coefficient function $\tau = 1$ in the following discussion.

Firstly, supposing the symmetry generator coefficient function $\xi = \xi(t)$, then we have $A = 0$, $B = \eta_v$ and $C = \eta_x$. In this case because the coefficients ξ, ϕ and η do not depend on v, we cannot obtain the nonlocal symmetry generator and new explicit solutions different from those obtained by using the classical symmetry of the adjoint equation (3).

Secondly, supposing the symmetry generator coefficient function $\xi = \xi(x)$, we have functions $A = 0$, $B = \eta_v - \xi_x$ and $C = \eta_x$. Substituting this into equations (9), it follows

$$\eta = 2\alpha_1(x,t)\exp\left(\frac{v}{2}\right) + \beta(x,t), \tag{10}$$

and

$$\begin{cases} \xi_{xx} - 2\xi\xi_x + \beta_x = 0, \\ 2(\alpha_{1t} - \alpha_{1xx} + 2\alpha_1\xi_x)\exp\left(\frac{v}{2}\right) = 0, \\ \beta_t - \beta_{xx} + 2\beta\xi_x = 0. \end{cases} \tag{11}$$

Integrating the first equation of equations (11) with respect to the variable x, the following three cases need to consider according to the different choice of the integrating constant.

Case A. Choose k^2 as an integral constant. Combining the first and third equations of equations (11), we can deduce

$$\begin{cases} \beta = \xi^2 - \xi_x + k^2, \\ \xi_{xxx} - 2\xi\xi_{xx} - 4\xi_x^2 + 2\xi^2\xi_x + 2k^2\xi_x = 0. \end{cases} \tag{12}$$

Solving equations (12), we can derive the following nontrivial special solutions:

$$\begin{cases} \xi_1 = a\tan a(x+c) - \dfrac{b}{2}, & \beta_1 = -ba\tan a(x+c), & \alpha_1 = \kappa\beta_1, \\ \xi_2 = 3a_1\tan a_1(x+c), & \beta_2 = 2a_1^2[3\tan^2 a_1(x+c) + 1], & \alpha_2 = \kappa\beta_2, \end{cases} \tag{13}$$

where $a = \frac{\sqrt{b^2+4k^2}}{2}$, $a_1 = \frac{k}{\sqrt{5}}$, b, c and κ are arbitrary constants.

Case B. With the choice of $-k^2$ as an integral constant, from equations (11), we can obtain

$$\begin{cases} \beta = \xi^2 - \xi_x - k^2, \\ \xi_{xxx} - 2\xi\xi_{xx} - 4\xi_x^2 + 2\xi^2\xi_x - 2k^2\xi_x = 0. \end{cases} \tag{14}$$

From equations (14), we acquire the following nontrivial special solutions:

$$\begin{cases} \tilde{\xi}_1 = a_2\tan a_2(x+c) - \dfrac{b_1}{2}, & \tilde{\beta}_1 = -b_1a_2\tan a_2(x+c), & \tilde{\alpha}_1 = \kappa\tilde{\beta}_1, \\ \tilde{\xi}_2 = -a_3\coth a_3(x+c) - \dfrac{b_2}{2}, & \tilde{\beta}_2 = b_2a_3\coth a_3(x+c), & \tilde{\alpha}_2 = \kappa\tilde{\beta}_2, \\ \tilde{\xi}_3 = -3a_1\tanh a_1(x+c), & \tilde{\beta}_3 = 2a_1^2[3\tanh^2 a_1(x+c) - 1], & \tilde{\alpha}_3 = \kappa\tilde{\beta}_3, \\ \tilde{\xi}_4 = -3a_1\coth a_1(x+c), & \tilde{\beta}_4 = 2a_1^2[3\coth^2 a_1(x+c) - 1], & \tilde{\alpha}_4 = \kappa\tilde{\beta}_4, \\ \tilde{\xi}_5 = \dfrac{1}{c_0 - x} \pm k, & \tilde{\beta}_5 = \pm\dfrac{2k}{c_0 - x}, & \tilde{\alpha}_5 = \kappa\tilde{\beta}_5. \end{cases} \tag{15}$$

Here $|b_1| \geqslant 2k$, $|b_2| \leqslant 2k$, $a_2 = \frac{\sqrt{b_1^2 - 4k^2}}{2}$, $a_3 = \frac{\sqrt{4k^2 - b_2^2}}{2}$, and c_0 is arbitrary.

Case C. Setting the integral constant equal to zero, from equations (11), we can derive

$$\begin{cases} \beta = \xi^2 - \xi_x, \\ \xi_{xxx} - 2\xi\xi_{xx} - 4\xi_x^2 + 2\xi^2\xi_x = 0. \end{cases} \tag{16}$$

By equations (16), we can derive the following nontrivial special solutions:

$$\begin{cases} \bar{\xi}_1 = \frac{b_3}{2}\left(\tan\frac{b_3}{2}(x+c) - 1\right), & \bar{\beta}_1 = -\frac{b_3^2}{2}\tan\frac{b_3}{2}(x+c), \quad \bar{\alpha}_1 = \kappa\bar{\beta}_1, \\ \bar{\xi}_2 = \frac{-3}{x+c}, & \bar{\beta}_2 = \frac{6}{(x+c)^2}, \quad \bar{\alpha}_2 = \kappa\bar{\beta}_2, \end{cases} \tag{17}$$

where b_3 is an arbitrary nonzero constant.

From (13), (15) and (17), we can derive nine sets of nonlocal symmetry generators of the Burgers equation. In what follows, we will consider explicit solutions from the nonlocal symmetry generators.

1. Setting the coefficient function $\xi = a\tan a(x+c) - \frac{b}{2}$, $\beta = \alpha = -ba\tan a(x+c)$, and substituting this into (10) and (7), we have

$$\begin{aligned} V_1 = {} & \frac{\partial}{\partial t} + \left[a\tan a(x+c) - \frac{b}{2}\right]\frac{\partial}{\partial x} - ba\tan a(x+c)\left(2\,\mathrm{e}^{\frac{v}{2}} + 1\right)\frac{\partial}{\partial v} \\ & - \left[ab\,\mathrm{e}^{\frac{v}{2}}(2a\sec^2 a(x+c) + u\tan a(x+c)) + a^2\sec^2 a(x+c)(b+u)\right]\frac{\partial}{\partial u}. \end{aligned} \tag{18}$$

Neither using the classical symmetry method nor using the nonclassical symmetry method is difficult to solve functions u, v directly from (18). We select

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{a\tan a(x+c) - \frac{b}{2}} = \frac{\mathrm{d}v}{-ba\tan a(x+c)\left(2\,\mathrm{e}^{\frac{v}{2}} + 1\right)}. \tag{19}$$

Solving equation (19), we have two independent integrals

$$\begin{cases} \zeta = t - \frac{2}{4a^2+b^2}[-bx + 2\ln|2a\sin a(x+c) - b\cos a(x+c)|], \\ \frac{-2b[2a^2x + b\ln|2a\sin a(x+c) - b\cos a(x+c)|]}{4a^2+b^2} + 2\ln\left(2 + \mathrm{e}^{-\frac{v}{2}}\right) = f(\zeta). \end{cases} \tag{20}$$

In order to construct explicit solutions, the unknown function $f(\zeta)$ can be determined by

$$\begin{cases} -\frac{v_t\,\mathrm{e}^{-\frac{v}{2}}}{2+\mathrm{e}^{-\frac{v}{2}}} = \frac{\partial\zeta}{\partial t}\frac{df}{d\zeta} = f', \\ \frac{-2ab\tan a(x+c)}{2a\tan a(x+c) - b} - \frac{v_x\,\mathrm{e}^{-\frac{v}{2}}}{2+\mathrm{e}^{-\frac{v}{2}}} = \frac{\partial\zeta}{\partial x}\frac{\mathrm{d}f}{\mathrm{d}\zeta} = f'\frac{-2}{2a\tan a(x+c) - b}, \\ \frac{2a^2b^2\sec^2 a(x+c)}{(2a\tan a(x+c) - b)^2} - \frac{\left(v_{xx} - \frac{v_x^2}{2}\right)\mathrm{e}^{-\frac{v}{2}}}{2+\mathrm{e}^{-\frac{v}{2}}} - \frac{1}{2}\left(\frac{v_x\,\mathrm{e}^{-\frac{v}{2}}}{2+\mathrm{e}^{-\frac{v}{2}}}\right)^2 = \frac{4f'' + 4a^2\sec^2 a(x+c)f'}{(2a\tan a(x+c) - b)^2}. \end{cases} \tag{21}$$

In view of the adjoint equation (3), from equations (21), we deduce $f(\zeta)$ satisfying

$$f'' + \frac{f'^2}{2} + \left(a^2 - \frac{b^2}{4}\right)f' - \frac{a^2b^2}{2} = 0. \tag{22}$$

The general solution of equation (22) is $f(\zeta) = \frac{b^2\zeta}{2} + 2\ln\left(-2\frac{c_2 - c_1\,\mathrm{e}^{-\frac{(4a^2+b^2)\zeta}{4}}}{4a^2+b^2}\right)$. Substituting this into equations (20), we arrive at

$$\begin{aligned} v = -2\ln\Big(\mathrm{e}^{\frac{b^2t+2bx}{4}}\Big(2c_1\,\mathrm{e}^{-\frac{(4a^2+b^2)t+2bx}{4}}(2a\cos a(x+c) - b\sin a(x+c)) \\ - 2c_2\Big) - 2(4a^2+b^2)\Big) + 2\ln(4a^2+b^2). \end{aligned} \tag{23}$$

From (23), we calculate the derivative of v with respect to x

$$v_x = \frac{e^{\frac{b^2t+2bx}{4}}\left[2ac_1\,e^{-\frac{(4a^2+b^2)t+2bx}{4}}(b\sin a(x+c)+2a\sin a(x+c))-bc_2\right]}{c_2\,e^{\frac{b^2t+2bx}{4}}-c_1\,e^{-a^2t}(2a\sin a(x+c)-b\cos a(x+c))+4a^2+b^2}. \tag{24}$$

It is easy to verify that (24) is a new explicit solution of the Burgers equation (1).

2. With the choice of $\xi = 3a_1\tan a_1(x+c)$ and $\beta = \alpha = 2a_1^2(1+3\tan^2 a_1(x+c))$, substituting these into (10) and (7), we derive

$$\begin{aligned} V_2 = {} & \frac{\partial}{\partial t} + 3a_1\tan a_1(x+x_0)\frac{\partial}{\partial x} + 2a_1^2(1+3\tan^2 a_1(x+c))\left(2\exp\left(\frac{v}{2}\right)+1\right)\frac{\partial}{\partial v} \\ & + \left[(12a_1\tan a_1(x+c)\sec^2 a_1(x+x_0) + (1+3\tan^2 a_1(x+c))u)2a_1^2\exp\left(\frac{v}{2}\right)\right. \\ & \left. + (4a_1\tan a_1(x+x_0)-u)3a_1^2\sec^2 a_1(x+x_0)\right]\frac{\partial}{\partial u}. \end{aligned} \tag{25}$$

From the nonlocal symmetry generator (25), we consider

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{3a_1\tan a_1(x+c)} = \frac{\mathrm{d}v}{2a_1^2(1+3\tan^2 a_1(x+c))\left(2\exp\left(\frac{v}{2}\right)+1\right)}. \tag{26}$$

Solving equation (26), we have

$$\begin{cases} \zeta_1 = t - \dfrac{1}{3a_1^2}\ln(\sin a_1(x+c)), \\ \dfrac{2}{3}\ln(\sin a_1(x+c)) - 2\ln(\cos a_1(x+c)) + 2\ln\left(2+\exp\left(-\dfrac{v}{2}\right)\right) = f(\zeta_1). \end{cases} \tag{27}$$

This time we can claim that the undetermined function $f(\zeta_1)$ satisfies

$$f'' + \frac{f'^2}{2} + 5a_1^2 f' + 8a_1^4 = 0. \tag{28}$$

Solving equation (28), we have $f(\zeta_1) = -8a_1^2\zeta_1 + 2\ln\left(c_1\,e^{3a_1^2\zeta_1} - c_2\right) + 2\ln\left(6a_1^2\right)$. Substituting $f(\zeta_1)$ into equations (27), we obtain

$$v = -2\ln\left(e^{-4a_1^2t}\cos a_1(x+c)\left(c_1\,e^{3a_1^2t} - c_2\sin a_1(x+c)\right) - 12a_1^2\right) + 2\ln\left(6a_1^2\right). \tag{29}$$

From (29), the derivative of v with respect to x is

$$v_x = 2\frac{a_1\,e^{-4a_1^2t}(\cos a_1(x+c)c_1\,e^{3a_1^2t} + c_2\cos 2a_1(x+c))}{c_1\,e^{-a_1^2t}\cos a_1(x+c) - c_2\,e^{-4a_1^2t}\cos a_1(x+c)\sin a_1(x+c)) - 12a_1^2}, \tag{30}$$

which is a new explicit solution of equation (1).

3. With the choice of $\xi = -3a_1\tanh a_1(x+c)$ and $\beta = \alpha = 2a_1^2[3\tanh^2 a_1(x+c) - 1]$, we derive

$$\begin{aligned} V_3 = {} & \frac{\partial}{\partial t} - 3a_1\tanh a_1(x+c)\frac{\partial}{\partial x} + 2a_1^2[3\tanh^2 a_1(x+c) - 1]\left(2\,e^{\frac{v}{2}}+1\right)\frac{\partial}{\partial v} \\ & + \left[2a_1^2\,e^{\frac{v}{2}}(12a_1\tanh a_1(x+c)\mathrm{sech}^2 a_1(x+c) + u(3\tanh a_1(x+c) - 1))\right. \\ & \left. + 3a_1^2\mathrm{sech}^2 a_1(x+c)(4a_1\tanh a_1(x+c) + u)\right]\frac{\partial}{\partial u}. \end{aligned} \tag{31}$$

Similarly, from the symmetry generator (31), we consider

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{-3a_1\tanh a_1(x+c)} = \frac{\mathrm{d}v}{2a_1^2(3\tanh^2 a_1(x+c) - 1)\left(2\,e^{\frac{v}{2}}+1\right)}. \tag{32}$$

Solving equation (32), we have

$$\begin{cases} \zeta_3 = t + \dfrac{1}{3a_1^2}\ln(\sinh a_1(x+c)), \\ \dfrac{2}{3}\ln(\sinh a_1(x+c)) - 2\ln(\cosh a_1(x+c)) + 2\ln\left(2+\mathrm{e}^{-\frac{v}{2}}\right) = f(\zeta_3). \end{cases} \tag{33}$$

This time the undetermined function $f(\zeta_3)$ satisfies

$$f'' + \frac{f'^2}{2} - 5a_1^2 f' + 8a_1^4 = 0. \tag{34}$$

The general solution is $f(\zeta_3) = 2a_1^2\zeta_3 + 2\ln\left(c_1\,\mathrm{e}^{3a_1^2\zeta_3} - c_2\right) - 2\ln\left(6a_1^2\right)$. Combining this with equations (33), it follows

$$v = -2\ln\left(\mathrm{e}^{a_1^2 t}\cosh a_1(x+c)\left(c_1\,\mathrm{e}^{3a_1^2 t}\sinh a_1(x+c) - c_2\right) - 12a_1^2\right) + 2\ln\left(6a_1^2\right). \tag{35}$$

From (35), the derivative of v with respect to x is

$$v_x = 2\frac{a_1\,\mathrm{e}^{a_1^2 t}\left(c_1\,\mathrm{e}^{3a_1^2 t}(1-\cosh^2 a_1(x+c)) + c_2\sinh a_1(x+c)\right)}{c_1\,\mathrm{e}^{4a_1^2 t}\cosh a_1(x+c)\sinh a_1(x+c) - c_2\,\mathrm{e}^{a_1^2 t}\cosh a_1(x+c)) - 12a_1^2}, \tag{36}$$

which is a new explicit solution of equation (1).

4. With the choice $\xi = -3a_1\coth a_1(x+c)$ and $\beta = \alpha = 2a_1^2[3\coth^2 a_1(x+c) - 1]$, we have

$$\begin{aligned} V_4 = {} & \frac{\partial}{\partial t} - 3a_1\coth a_1(x+c)\frac{\partial}{\partial x} + 2a_1^2[3\coth^2 a_1(x+c) - 1]\left(2\,\mathrm{e}^{\frac{v}{2}}+1\right)\frac{\partial}{\partial v} \\ & + \left[2a_1^2\,\mathrm{e}^{\frac{v}{2}}(-12a_1\coth a_1(x+c)\mathrm{csch}^2 a_1(x+c) + u(3\coth a_1(x+c) - 1)) \right. \\ & \left. - 3a_1^2\mathrm{csch}^2 a_1(x+c)(4a_1\coth a_1(x+c) + u)\right]\frac{\partial}{\partial u}. \end{aligned} \tag{37}$$

From the symmetry generator (37), we think of the following characteristic equation:

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{-3a_1\coth a_1(x+c)} = \frac{\mathrm{d}v}{2a_1^2(3\coth^2 a_1(x+c) - 1)\left(2\,\mathrm{e}^{\frac{v}{2}}+1\right)}. \tag{38}$$

Solving equation (38), we obtain

$$\begin{cases} \zeta_4 = t + \dfrac{1}{3a_1^2}\ln(\cosh a_1(x+c)), \\ \dfrac{2}{3}\ln(\cosh a_1(x+c)) - 2\ln(\sinh a_1(x+c)) + 2\ln\left(2+\mathrm{e}^{-\frac{v}{2}}\right) = f(\zeta_4). \end{cases} \tag{39}$$

The undetermined function $f(\zeta_4)$ satisfies

$$f'' + \frac{f'^2}{2} - 5a_1^2 f' + 8a_1^4 = 0. \tag{40}$$

The general solution is $f(\zeta_4) = 2a_1^2\zeta_4 + 2\ln(c_1\,\mathrm{e}^{3a_1^2\zeta_4} - c_2) - 2\ln\left(6a_1^2\right)$. Substituting $f(\zeta_4)$ into equations (39), we obtain

$$v = -2\ln\left(\mathrm{e}^{a_1^2 t}\sinh a_1(x+c)(c_1\,\mathrm{e}^{3a_1^2 t}\cosh a_1(x+c) - c_2) - 12a_1^2\right) + 2\ln\left(6a_1^2\right). \tag{41}$$

From (41), the derivative of v with respect to x is

$$v_x = 2\frac{a_1\,\mathrm{e}^{a_1^2 t}\left(c_1\,\mathrm{e}^{3a_1^2 t}(1-\cosh^2 a_1(x+c)) + c_2\cosh a_1(x+c)\right)}{c_1\,\mathrm{e}^{4a_1^2 t}\cosh a_1(x+c)\sinh a_1(x+c) - c_2\,\mathrm{e}^{a_1^2 t}\sinh a_1(x+c)) - 12a_1^2}, \tag{42}$$

which is a new explicit solution of equation (1).

5. With the choice of $\xi = -a_3 \coth a_3(x+c) - \frac{b_2}{2}$, and $\beta = \alpha = b_2 a_3 \coth a_3(x+c)$, we derive the following nonlocal symmetry generator:

$$V_5 = \frac{\partial}{\partial t} - \left[a_3 \coth a_3(x+c) + \frac{b_2}{2}\right]\frac{\partial}{\partial x} + a_3 b_2 \coth a_3(x+c)\left(2\,\mathrm{e}^{\frac{v}{2}}+1\right)\frac{\partial}{\partial v} + \big[a_3 b_2\,\mathrm{e}^{\frac{v}{2}}(u\coth a_3(x+c) - a_3 \mathrm{csch} a_3(x+c)) - a_3^2 \,\mathrm{csch}^2 a_3(x+c)(b_2+3u)\big]\frac{\partial}{\partial u}. \tag{43}$$

From the symmetry generator (43), we think of the characteristic equation

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{-a_3\coth a_3(x+c) - \frac{b_2}{2}} = \frac{\mathrm{d}v}{a_3 b_2 \coth a_3(x+c)\left(2\,\mathrm{e}^{\frac{v}{2}}+1\right)}. \tag{44}$$

Solving equation (44), we have

$$\begin{cases} \zeta_5 = t + \dfrac{2}{b_2^2 - 4a_3^2}(b_2 x - 2\ln(|2a_3\cosh a_3(x+c) + b_2 \sinh a_3(x+c))), \\ \dfrac{2b_2}{b_2^2-4a_3^2}\left(2a_3^2 x - b_2 \ln\left(|2a_3\cosh a_3(x+c) + b_2\sinh a_3(x+c)\right)\right) \\ \quad +2\ln\left(2+\mathrm{e}^{-\frac{v}{2}}\right) = f(\zeta_5). \end{cases} \tag{45}$$

This time we can deduce the undetermined function $f(\zeta_5)$ which satisfies

$$f'' + \frac{f'^2}{2} - \left(a_3^2 + \frac{b_2^2}{4}\right) f' + \frac{a_3^2 b_2^2}{2} = 0. \tag{46}$$

Solving equation (46), we have $f(\zeta_5) = \frac{b_2^2}{2}\zeta_5 + 2\ln\left(2c_2 - 2c_1\,\mathrm{e}^{\frac{4a_3^2-b_2^2}{4}\zeta_5}\right) - 2\ln\left(4a_3^2 - b_2^2\right)$. Substituting $f(\zeta_5)$ into equations (45), we obtain

$$v = -2\ln\left(\mathrm{e}^{\frac{b_2^2 t + 2b_2 x}{4}}\left(2c_2 - 2c_1\,\mathrm{e}^{\frac{(4a_3^2-b_2^2)t - b_2 x}{4}}(2a_3\sinh a_3(x+c) + b_2\cosh a_3(x+c))\right) - 8a_3^2 + 2b_2^2\right) + 2\ln\left(4a_3^2 - b_2^2\right). \tag{47}$$

From (47), the derivative of v with respect to x is

$$v_x = -\frac{1}{2}\frac{\mathrm{e}^{\frac{b_2^2 t+2b_2 x}{4}}\left(-c_2 b_2^2 + 4c_1 a_3^2\,\mathrm{e}^{\frac{(4a_3^2-b_2^2)t-b_2 x}{4}}(2a_3\cosh a_3(x+c) + b_2\sinh a_3(x+c))\right)}{c_1\,\mathrm{e}^{a_3^2 t}(2a_3\cosh a_3(x+c) + b_2\sinh a_1(x+c)) - c_2\,\mathrm{e}^{\frac{b_2^2 t+2b_2 x}{4}} + 4a_3^2 - b_2^2}. \tag{48}$$

This is a new explicit solution of the equation (1).

6. With the choice $\xi = \frac{-3}{x+c}$ and $\beta'' = \alpha'' = \frac{6}{(x+c)^2}$, we acquire

$$V_6 = \frac{\partial}{\partial t} - \frac{3}{x+c}\frac{\partial}{\partial x} + \frac{6}{(x+c)^2}\left(2\,\mathrm{e}^{\frac{v}{2}}+1\right)\frac{\partial}{\partial v} + \left[\frac{6}{(x+c)^2}\mathrm{e}^{\frac{v}{2}}\left(u - \frac{4}{x+c}\right) - \frac{3}{(x+c)^2}\left(u + \frac{4}{x+c}\right)\right]\frac{\partial}{\partial u}. \tag{49}$$

From the symmetry generator (49), we only consider the following characteristic equation:

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{\frac{-3}{x+c}} = \frac{\mathrm{d}v}{\frac{6}{(x+c)^2}\left(2\,\mathrm{e}^{\frac{v}{2}}+1\right)}. \tag{50}$$

Solving equation (50), we have

$$\begin{cases} \zeta_6 = t + \dfrac{(x+c)^2}{6}, \\ -2\ln(x+c) + 2\ln\left(2+\mathrm{e}^{-\frac{v}{2}}\right) = f(\zeta_6). \end{cases} \tag{51}$$

This time we know the unknown function $f(\zeta_6)$ which satisfies

$$f'' + \frac{f'^2}{2} = 0. \tag{52}$$

The general solution is $f(\zeta_6) = 2\ln(c_1\zeta_5 - c_2) - 2\ln 2$. Substituting $f(\zeta_6)$ into equations (51), we obtain

$$v = -2\ln\left((x+c)\left(c_1\left(t + \frac{(x+c)^2}{6}\right) + c_2\right) - 4\right) + 2\ln 2. \tag{53}$$

From (53), the derivative of v with respect to x is

$$v_x = -2\frac{c_1\left(t + \frac{(x+c)^2}{2}\right) + c_2}{(x+c)\left(c_1\left(t + \frac{(x+c)^2}{6}\right) + c_2\right) - 4}. \tag{54}$$

This is also an explicit solution of equation (1).

7. With the choice of $\xi = \frac{1}{c_0 - x} + k$ and $\beta'' = \alpha'' = \frac{2k}{c_0 - x}$, we have

$$\begin{aligned} V_7 = {} & \frac{\partial}{\partial t} + \left(\frac{1}{c_0 - x} + k\right)\frac{\partial}{\partial x} + \frac{2k}{c_0 - x}\left(2\,\mathrm{e}^{\frac{v}{2}} + 1\right)\frac{\partial}{\partial v} \\ & + \left[\frac{2k}{c_0 - x}\mathrm{e}^{\frac{v}{2}}\left(u + \frac{2}{c_0 - x}\right) - \frac{1}{(c_0 - x)^2}(u - 2k)\right]\frac{\partial}{\partial u}. \end{aligned} \tag{55}$$

From the nonlocal symmetry generator (55), we only consider the following characteristic equation:

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{\frac{1}{c_0 - x} + k} = \frac{\mathrm{d}v}{\frac{2k}{c_0 - x}\left(2\,\mathrm{e}^{\frac{v}{2}} + 1\right)}. \tag{56}$$

Solving equations (56), we have

$$\begin{cases} \zeta_7 = t - \dfrac{1}{k} - \dfrac{1}{k^2}\ln(1 + k(c_0 - x)), \\ -2\ln(1 + k(c_0 - x)) + 2\ln\left(2 + \mathrm{e}^{-\frac{v}{2}}\right) = f(\zeta_7). \end{cases} \tag{57}$$

This time the unknown function $f(\zeta_7)$ satisfies

$$f'' + \frac{f'^2}{2} - k^2 f' = 0. \tag{58}$$

Solving equation (58), we have the general solution $f(\zeta_7) = 2\ln\left(c_1\,\mathrm{e}^{k^2\zeta_7} + c_2k^2\right) - 2\ln k^2$. Substituting $f(\zeta_7)$ into equations (57), we obtain

$$v = -2\ln\left(c_1\,\mathrm{e}^{k^2t - kx} + c_2k^2(k(c_0 - x) + 1) - 4k^2\right) + 2\ln k^2. \tag{59}$$

From (59), the derivative of v with respect to x is

$$v_x = 2\frac{kc_1\,\mathrm{e}^{k^2t - kx} + c_2k^3}{c_1\,\mathrm{e}^{k^2t - kx} + c_2k^2(k(c_0 - x) + 1) - 4k^2}. \tag{60}$$

Another explicit solution of equation (1) is obtained.

Finally, choosing the symmetry generator coefficient function $\xi = \xi(x, t)$, we have functions $A = 0$, $B = \eta_v - \xi_x$ and $C = \eta_x$. Substituting these into equations (9), it follows that the coefficient function η is determined by (10) and

$$\begin{cases} \xi_t - \xi_{xx} + 2\xi\xi_x - \beta_x = 0, \\ 2(\alpha_{1_t} - \alpha_{1_{xx}} + 2\alpha_1\xi_x)\exp\left(\dfrac{v}{2}\right) = 0, \\ \beta_t - \beta_{xx} + 2\beta\xi_x = 0. \end{cases} \tag{61}$$

Introducing a new variable $z = x + k_3 t$ (k_3 is a nonzero constant), the first and the third equations of equations (61) can be transformed into

$$\begin{cases} k_3\beta = \xi^2 - \xi_z - k_3\xi + d, \\ \xi_{zzz} - 2\xi\xi_{zz} - 4\xi_z^2 + 2\xi^2\xi_z + (2d + k_3)\xi_z - (k_3 + 1)\xi_{zz} + 2(k_3 + 1)\xi\xi_z = 0, \end{cases} \tag{62}$$

where d is an integral constant. By choosing $k_3 = -1$, we only consider the following cases according to the choice of the integral constant d.

Choosing an integral constant $d = \frac{k^2+1}{2}$, and using (13), we can deduce the following nontrivial special solutions:

$$\begin{cases} \hat{\xi}_1 = a\tan a(x-t) - \dfrac{b}{2}, & \hat{\beta}_1 = (b-1)a\tan a(x-t) + \dfrac{b-1+k^2}{2}, \\ \hat{\xi}_2 = 3a_1\tan a_1(x-t), & \hat{\beta}_2 = 3a_1\tan a_1(x-t)[2a_1\tan a_1(x-t)+1] + \dfrac{k^2-5}{10}, \end{cases} \tag{63}$$

where the constants a, b and a_1 are the same as in (13), function $\hat{\alpha}_i = \kappa\hat{\beta}_i$ for $i = 1, 2$.

With the choice of $d = \frac{1-k^2}{2}$ and using (15), we can obtain the following nontrivial special solutions:

$$\begin{cases} \check{\xi}_1 = a_2\tan a_2(x-t) - \dfrac{b_1}{2}, & \check{\beta}_1 = (b_1-1)a_2\tan a_2(x-t) + \dfrac{b_1-1-k^2}{2}, \\ \check{\xi}_2 = -\,a_3\coth a_3(x-t) - \dfrac{b_2}{2}, & \check{\beta}_2 = (1-b_2)a_3\coth a_3(x-t) + \dfrac{b_2-1-k^2}{2}, \\ \check{\xi}_3 = -\,3a_1\tanh a_1(x-t), & \check{\beta}_3 = 3a_1\tanh a_1(x-t)[2a_1\tanh a_1(x-t)+1] - \dfrac{5k^2+1}{2}, \\ \check{\xi}_4 = -\,3a_1\coth a_1(x-t), & \check{\beta}_4 = 3a_1\tanh a_1(x-t)[-2a_1\tanh a_1(x-t)+1] - \dfrac{5k^2+1}{2}, \\ \check{\xi}_5 = \dfrac{1}{-x+t} \pm k, & \check{\beta}_5 = \dfrac{\pm 2k-1}{-x+t} - \dfrac{(k\pm 1)^2}{2}, \end{cases} \tag{64}$$

Here, the constants a_1, a_2, a_3, b_1, b_2 are similar to those that appeared in (15), function $\check{\alpha}_i = \kappa\check{\beta}_i$ for $i = 1, 2, \ldots, 5$.

Setting the integral constant $d = \frac{1}{2}$, it follows

$$\begin{cases} \check{\xi}_1 = \dfrac{b_3}{2}\left(\tan\dfrac{b_3}{2}(x-t) - 1\right), & \check{\beta}_1 = \dfrac{b_3}{2}\tan\dfrac{b_3}{2}(x-t) + \dfrac{b_3-1}{2}, \\ \check{\xi}_2 = \dfrac{-3}{x-t}, & \check{\beta}_2 = -\dfrac{6}{(x-t)^2} + \dfrac{3}{x-t} - \dfrac{1}{2}, \end{cases} \tag{65}$$

where b_3 is an arbitrary nonzero constant, and functions $\check{\alpha}_i = \kappa\check{\beta}_i$ for $i = 1, 2$.

From (63)–(65), we can derive nine sets of nonlocal symmetry generators. To our knowledge, these nonlocal symmetry generators have not been found in previous literatures. It is difficult for us to construct explicit solutions from these nine sets of nonlocal symmetry generators by repeating the former procedure. Explicit solutions obtained from these new symmetry generators need further investigation.

3. Conclusion

In summary, the nonlocal symmetry and explicit solutions of a partial differential equation, which can be written as a conservation law, are considered in this paper. The Burger equation is used as an example to illustrate details. Nineteen sets of nonlocal symmetry generators and many new explicit solutions of the Burgers equation are obtained. Many results have not been touched in previous literatures. Just like Boiti *et al* [14] state, finding nonlocal symmetries is crucial to find not only how the original dependent variable (such as u in the paper) changes, but also how the new dependent variable (like v in the present paper) changes along the symmetry. This is precisely demonstrated by the work of this paper. Although the solutions obtained in this paper satisfy the famous Cole–Hopf transformation, it is really difficult for us to obtain them directly from the heat equation via the Cole–Hopf transformation. The method which can be used to construct an explicit solution from nonlocal symmetry generators (63)–(65) is worth further study.

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